Audrey's acquisition of fractions: a case study into the learning of formal mathematics

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AUDREY'S ACQUISITION OF FRACTIONS: A CASE STUDY INTO THE LEARNING OF FORMAL MATHEMATICS

ABSTRACT. National standards for teaching mathematics in primary schools in the Netherlands leave little room for formal fractions. However, a newly developed programme in fractions aims at learning formal fractions. The starting point in the development of this curriculum is the students' acquisition of 'numeracy in fractions'. In this case study we describe the growth in reasoning ability with fractions of one student in this newly developed programme of 30 lessons during one whole school year. In the study we found indications that the programme and its teaching stimulated the progress of an average performer in mathematics. Moreover we found arguments as to what extent formal operations with fractions suits as an educational goal.

KEY WORDS: case study, curriculum development, formalising, fractions, modelling, numeracy, primary education, realistic mathematics education, representations

1. INTRODUCTION

Students differ in many aspects. Restricted to the learning of fractions, one observes how some ten year olds relatively easily acquire rational numbers, while others experience difficulties in the most simple manipulations with fractions. Many researchers reported on the differences in students' fraction learning (cf. e.g. Holt, 1964; Behr, Lesh, Post and Silver, 1983; Hunting, 1984; Streefland, 1987; Behr, Harel, Post and Lesh, 1992; Kamii and Clark, 1995). In addition, the Dutch National Testing Institute for primary and secondary education researches the development of the mathematical ability of students in grade 8 in primary schools (11–12 year olds). The three nationwide investigations performed so far show, again and again, tremendous differences in skills between students, especially in fractions (Wijnstra, 1988; Bokhove, Van der Schoot and Eggen, 1996; Janssen, Van der Schoot, Hemker and Verhelst, 1999). These investigations suggest that a significant number of students should be able to acquire the ability for formal reasoning with fractions rather easily.

However, national curriculum standards for primary education establish that teaching should be aimed at acquiring competence in using fractions in simple contexts or supported by models (Commissie Heroverweging Kerndoelen Basisonderwijs [Committee for the Reassessment of Cur-

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riculum Standards in Primary Education], 1994; cf. Principles and Standards for School Mathematics, 2000). We thus ascertain a discrepancy between the potentialities of groups of students and what is decided upon as curriculum standards. This discrepancy forms the starting point of the research described here. In the Dutch tradition of realistic mathematics education (Treffers, 1987; Freudenthal, 1991; Streefland, 1991; Gravemeijer, 1994; Van den Heuvel-Panhuizen, 1996), we developed an experimental curriculum on fractions as an extension of the newly developed curriculum, 'the Fractiongazette' [De Breukenbode] (Buys, Bokhove, Keijzer, Lek, Noteboom and Treffers, 1996). This experimental curriculum is constructed to seize upon potentialities of students who are able to obtain meaningful formal reasoning with fractions with relative ease, where we will refer to this formal level in reasoning as 'formal fractions'. This level of reasoning "is characterised by a sense that one's mathematical methods work 'for all' relevant examples" (Pirie and Kieren, 1994, 43). Or, following Hart (1987), we consider this "formalisation" to be "(...) a rule, formula or general method which can be applied to a variety of mathematical examples" (p. 409). In the study reported on here, we will observe a process of learning formal mathematics. Freudenthal (1973) considers this to be a process where a general principle emerges from a series of well chosen examples. Here we will report upon this process leading to the generalised notion of equivalent fractions, which on their turn facilitate fraction operations.

During one school year we observed the development of one of the students, Audrey, in the experimental programme. We describe her fraction learning here as a case-study, and analyse Audrey's development in learning fractions. Audrey is a student with average skills in mathematics.[1] Elsewhere we report on the development of students involved in the experiments in a quasi-experimental research design (Keijzer and Terwel, 2000). Our analyses here finally lead to assessing general characteristics on learning meaningful formal fractions.

2. Designing a Programme

Mathematical insight is widely recognised as an important educational goal. Mathematics education should promote learning for understanding (Freudenthal, 1968; Van Hiele, 1986; Sfard, 1994; Perkins and Unger, 1999; Reigeluth, 1999; Van Dijk, Van Oers and Terwel, 2000). However, it is known from many studies that students have difficulty in applying their mathematical knowledge in meaningful ways in formal mathematics. Moreover history proves that teaching mathematics often results in imitative, meaningless following of rules of calculation. This is especially so for the learning of fractions.

For this reason, various researchers in the 1980’s and 1990’s pleaded for constructivist approaches to the difficult problem of teaching fractions. For example, Graeber and Tanenhaus (1993) suggest naming fractions by making them results of measuring. With Bednarz and Janvier (1988) and Mack (1990), Graeber and Tanenhaus chose to explicitly build on informal knowledge of students.[2]

In the Netherlands, Streefland (1982, 1983, 1987 and 1990) developed a new curriculum on fractions in the Dutch tradition of realistic mathematics education. By choosing fair-sharing as main fraction-generating activity, Streefland also took informal knowledge of the students into account. Through these activities of fair-sharing, Streefland at first stimulated the development of a fraction-language by the students. For example students develop their language of fractions, when they have to divide three pizzas among four children. Each child gets three pieces of a quarter of a pizza or half a pizza and a quarter of a pizza, etc. Later Streefland elaborated the sharing-situation so that equivalent fractions could emerge; equivalent sharing situations were observed. Sharing three pizzas among four children results in the same amount of pizza for each child as sharing six pizzas among eight. Moreover, by comparing results of fair-sharing formal operations with fractions are facilitated.

As we mentioned, in the late 1990’s a new fraction programme, ‘The Fractiongazette’ [De Breukenbode] was developed (Buys, Bokhove, Keijzer, Lek, Noteboom and Treffers, 1996). This programme was created to link up with new curriculum standards for teaching fractions. Moreover, it provided explicitly for students’ acquisition of ‘numeracy’ or number sense (Greeno, 1991; Mcintosh, Reys and Reys, 1992; Keijzer and Buys, 1996a). Our programme, which we describe here and which, for convenience, we name an ‘experimental programme’, is an extension of ‘The Fractiongazette’, and emphasises formal reasoning with fractions.

As in Streefland’s curriculum, in the experimental programme at first there is explicit focus on the learning of fraction-language. However, unlike Streefland, in this programme situations of measuring are mainly used. Bezuk and Bieck (1993) emphasise the importance of this kind of situation in teaching fractions; in this manner fractions are seen as lengths, which help students in making estimations and thus facilitates reflection on one’s work. Connell and Peck (1993) provide arguments for using a bar as a measuring instrument as a forerunner for the number line. They observed students’ preference for the bar (in the context of a rectangular cake) as a model.

> The students (. . .) universally selected a 'cake' model for dealing with fractions because they seemed to sense its general applicability.
>
> (Connell and Peck, 1993, p. 336)

In the next stages of the experimental programme, from the situations of measuring, the number line is developed. Moreover, this model forms the key-instrument in comparing fractions. Many strategies for comparing fractions are discussed with the students. Furthermore, the situations presented encourage students to use equivalent fractions more and more when comparing. Next these equivalent fractions form a base for formal operations with fractions.

Several researchers (e.g. Behr, Lesh, Post and Silver (1983) and Novillis Larson (1980)) reported on students' difficulties with the number line. Behr, Lesh, Post and Silver observed three kinds of problems:

> 1. Children differed in how they identified the unit on the number line.
> 2. Problems, in which the subdivisions of the unit did not equal the denominator of the fraction, were harder to solve than were problems in which subdivisions equalled the denominator.
> 3. Problems with perceptual distractors (inconsistent cues) were harder to solve than were problems in which subdivisions of the unit were factors or multiples of the denominator or the fraction (incomplete cues or irrelevant cues).
>
> (Behr, Lesh, Post and Silver, 1983, p. 118)

To overcome these problems, in the experimental curriculum using a bar as a measuring-device, and then making measurements was closely connected with the development of the number line. Moreover, the number line was used extensively to compare fractions (Keijzer and Buys, 1996b). The following observation by Mack (1990) provides additional support for relating the development of the number line and comparing fractions. Students, she observed, spontaneously find and use a comparison-strategy, that clearly is supported by the number line:

> One common characteristic of all student-invented algorithms, with the exception of the alternative algorithm for comparing fractions (via 1, RK/JT), was that in general, they were not utilised for an extended period of time. Students soon discovered quicker ways of solving the problems. As soon as they discovered these quicker algorithms, they abandoned their alternative algorithms in favour of the more efficient ones, which often reflected those that are traditionally taught in schools. (Mack, 1990, p. 26–27)

In the final stage of the programme formal operations with fractions become a field of exploration for the students (cf. Streefland and Elbers, 1995 and 1997). However, as formal fractions are difficult (Hart, 1981; Hasemann, 1981; Hiebert, 1988; Kamii and Clark, 1995), in the experimental program formal approaches were used next to informal ones. Moreover, dealing with formal and informal fractions occured next to each other, intending to facilitate switching from one strategy to another.

The finalised programme thus had the following key features:

- As whole class discussions are used to negotiate and construct meanings in learning fractions, the teaching can be characterised as interactive.
- The curriculum is directed towards the acquisition of number sense: students learn to give meaning to fractions in various kinds of situations, develop a good notion of the size of fractions, and learn to handle fractions in simple applications.
- The curriculum contains a teaching strategy in four stages in which number sense is developed: (i) a language of fractions, (ii) developing the number line for fractions, (iii) comparing fractions, (iv) learning formal fractions.
- Different situational contexts and models are used: two types of situations, dividing and measuring, lead to the bar and the number line as central models for fractions.
- Students are offered the opportunity to present their approaches at several levels: initially, when confronted with fraction problems, they opt for informal approaches. These are followed by semi-formal and formal solutions, which are imbedded in the informal approaches. Thus the students are challenged to reach approaches at higher levels.

3. AIM

Our objective is to describe, analyse and explain the complex fraction learning process of an average student, Audrey. Yin (1984) clearly considers a case study appropriate here, as the study is sustained by a theoretical framework. Moreover, he states on case studies:

> "The most important [thing] is to *explain* the causal links in real-life interventions that are too complex for the survey or experimental strategies. A second application is to *describe* the real-life context in which an intervention has occurred. Third, an evaluation can benefit, again in a descriptive mode, from an illustrative case study (...) of the intervention itself. Finally, the case study strategy may be used to *explore* those situations in which the intervention being evaluated has no clear, single set of outcomes. (Yin, 1984, p. 25)

Yin thus offers us a methodology to design the case-study. We describe Audrey's fraction learning process in the newly developed programme to

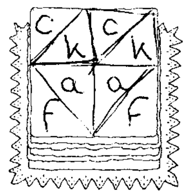

Figure 1. Audrey's cake with four toppings.

obtain a description of the real-life context where the intervention took place. Moreover, Greeno (1991) offers us a theoretical framework that, together with the theoretical notions mentioned, we will use to analyse and explain the relation between the fraction-programme, the style of teaching and Audrey's strategies in handling formal and informal fractions. We developed what Greeno refers to as an "environment that fosters curiosity and exploration" (p. 173) and will analyse relevant relations within this environment; those between teaching and learning fractions, between the programme and the teaching thereof and between activities in the programme and Audrey's fraction learning process. We will show relevant elements of the developed environment and will show how Audrey's learning is supported by a process of 'negotiation of meaning'.

Against this background we will argue how students with average mathematical skills can acquire formal fractions, in a programme that aims at students gaining number sense and which reaches formal approaches in situations where the number line is a central model and where comparing fractions is a key-activity.

4. AUDREY'S LEARNING OF FRACTIONS

4.1. *Obtaining a language of fractions*

Many researchers emphasise the importance of gaining competence in a language of fractions (Bezuk and Bieck, 1993; Connell and Peck, 1993; Streefland, 1990). This acquisition of fraction language therefore marks the beginning of the fraction program discussed here. Audrey's first lessons in fractions aim at dividing square-shaped and circle-shaped objects in parts of equal size. Moreover the pieces are named in both informal and formal manners and are symbolised as unit-fractions. For instance in the first lesson we ask Audrey to make a square cake with four different toppings. She does so with remarkable ease (Figure 1).

In the next lessons Audrey quickly starts to use unit-fractions in a correct manner. Then, in the fourth lesson a new context is presented to introduce other than unit fractions. We ask the students to measure their table with a bar, representing an 'Amsterdam foot' (abbreviated to av). Folding the bar leads to accurate measuring results, but generates the problem of naming the pieces of the bar. At first students use informal and long names for the fractions that arise, like a quarter or three pieces of the bar that is folded in eight. Audrey soon shortens these fraction-names to the formal notation.

This situation shows what Greeno (1991) refers to as a "a social construction in which students interact with the teacher and with each other about quantities and numbers" (p. 173). Here the meaning of fraction is 'negotiated' as precise names are needed to communicate measuring results.

However, Audrey does not yet fully understand the fraction-language. In the fourth lesson she also introduces 1/3 av as alternative name for three quarters av. However, Audrey continuously uses correct interpretations of the fractions and gradually improves her names and notations of the fractions.

After ten lessons we interviewed Audrey, to assess her knowledge of the language of fractions. One of the problems we present her here is about Irene's chocolate bar. We tell Audrey Irene ate 3/5 of her chocolate bar and show her what is left.

In solving this problem, Audrey shows her mastery of fraction language:

Interviewer:	Do you think Irene ate more than half the bar?
Audrey:	Yes...
I:	Can you explain?
A:	The bar has five pieces and she ate three. Two are left. (indicates two pieces in the drawn part of the bar)
I:	Can you name these pieces?
A:	Eh. . .one-second. . . Oh no, that is wrong. . .
I:	Can you draw the whole chocolate bar?
A (after drawing the bar):	Three pieces are out and two left.

Audrey now knows how to name the pieces as fractions. She writes what part is left (Figure 2).

We thus observe how Audrey, in about ten lessons on fractions, obtains a firm grip on the language involved in working with fractions. Her experiences in dividing objects and measuring with divided bars and, moreover, discussing the outcomes of these activities, resulted in the development of the use of fractions as descriptors in this kind of situation.

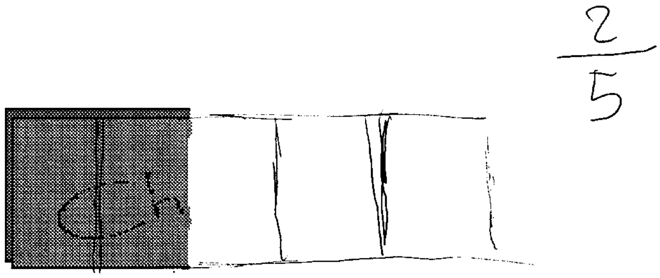

Figure 2. Audrey's solution of the Irene-problem.

4.2. *Developing the number line for fractions*

In the experimental programme, the number line is an important tool to reach formal (operations with) fractions. Equivalent fractions emerge as fractions in the same position on the line. Moreover, introducing the number line here enables us to exploit students' knowledge of operations with whole numbers and stimulates students to make rough calculations with fractions, such as 3/4 + 1/3 makes approximately 1. However, it takes students some effort to grasp the number line (Novillis Larson, 1980). Measuring with the 'Amsterdam foot' is one of the activities to help students to develop the number line for fractions. In general the bar, as a model for fractions, can be seen as means to form the number line.

In the following weeks the bar is further developed as a model for fractions. Sometimes the connection with the number line is made explicit. In lesson eleven we introduce the try-your-strength machine (Noteboom, 1994). If one hits the machine, water starts running through a pipe. When the water in the machine reaches the top, you are the strongest of all. Many children however, cannot reach the top. Audrey here in a free production compares the efforts of two children hitting to 1/2 and 1/3. Audrey, in comparing these results, moves her finger alongside the water pipe and concludes that 1/2 is higher than 1/3. Her bar has flattened to a line (Figure 3).

In the thirteenth lesson we use a drawing-contest as context. Approximately 600 children are participating in the contest. 1/5 of those are in the youngest (4 and 5 year) group. We ask the students, among other things, how many children in the contest were 4 or 5 years old. To solve this problem, we suggest the students to represent the 600 participants in a bar. Audrey, in doing so, more or less constructs a double indexed number line, where the participants in the contest are on the one side and fractions on the other. Audrey explains what she did: "If you take this five times you arrive at 600." On the whole we observe that using a double indexed bar

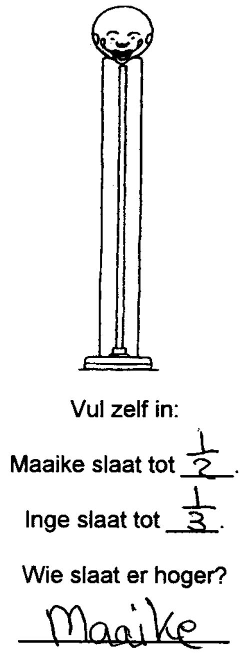

Figure 3. Audrey's free production with the try-your-strength machine.

or number line becomes Audrey's approach in solving problems, where a fraction is operating on a large number such as 600.

4.3. *Comparing fractions*

In the following lessons the number line is developed further. We thus encourage students to form various strategies to compare fractions to finally reach equivalent fractions (Keijzer and Buys, 1996b). These equivalent fractions provide a way to compare fractions in an algorithmic manner, for instance by transforming the two fractions involved in equivalents, that have an equal numerator or denominator. Moreover equivalent fractions are key to formal reasoning with fractions.

However, considerable effort is needed to grasp equivalent fractions in order to use them in the described way. In one of the first lessons, lesson 6, we observe how Audrey misinterprets the equivalence of fractions. In the context of the 'Amsterdam foot' we ask the students to compare the heights of Christel, 2 1/3 av, and Tessa, 2 2/6 av. Audrey thinks that Christel is taller.

In the same lesson we find a clue for this comparison strategy. Audrey compares with ease the lengths of Melle, 6 1/3 av, and Auke, 6 1/4 av. In explaining her approach Audrey points at the lengths of the pieces 1/3 and

1/4 and thus concludes that Auke is a bit shorter. We think that Audrey, in comparing fractions, at this stage of her fraction learning, concentrates on the denominator only. The larger the denominator the smaller the pieces. Reasoning this way a larger denominator always leads to a smaller fraction, independently of the size of the numerator of the fractions involved (cf. Noelting, 1980).

We saw how the context of the try-your-strength machine in the eleventh lesson again resulted in Audrey comparing fractions by looking at the size of the pieces. In the fifteenth lesson we introduced the context of the fraction-lift.[3] Here again we introduce a metaphor for learning fractions. Sfard (1994), from several interviews with mathematicians, shows how experts and novices use metaphors to construct mathematical knowledge. One of her interviewees tells her how he uses personification to perform manipulations on the concept (Sfard, 1994, p. 48). We see this personification in the context of the fraction-lift. Moreover, Greeno's (1991) notion of situated knowledge portrays the fraction-lift as interacting "with the environment in its own terms – exploring the territory, appreciating its scenery, and understanding how its various components interact." (p. 175).

Here a vertical number line houses fractions; the fractions live in a fraction-building. Lifts connect the different floors in the building. The numbers of the lifts indicate the stops they make: for instance the 3-lift stops three times, at 1/3, 2/3 and at the top of the building (at 1). Similarly the 4-lift stops at 1/4, 2/4, 3/4 and at 1, the 2-lift stops at 1/2 and at 1, etc. (Figure 4). This context thus makes explicit that different fractions can belong to the same position on the number line. In other words the fraction-lift, by personalising the fractions, becomes a metaphor for fractions on the numberline.

In the next lesson Audrey uses this newly introduced context, when comparing 3/10 and 3/13, to show how she extended her initial comparison strategy to fractions that are not unit-fractions. Moreover in this lesson we observe Audrey using 1 as anchor-point to compare fractions. The teacher and the students discuss finding fractions in the fraction-building higher than 99/100. One of the students mentions 199/200 and 299/300 as candidates. Audrey explains how this last result could be established: "Only 1/300 is needed to reach the top of the building."

In the nineteenth lesson equivalent fractions are again approached as fractions in the same position on the number line. In the lesson we discuss with the students which fractions occupy the same place on the line as 2/3. Many students here choose to (repeatedly) double both numerator and denominator of the fraction to make equivalent fractions (cf. Streefland, 1990). Audrey thus constructs the fraction 16/24. Next we discuss how

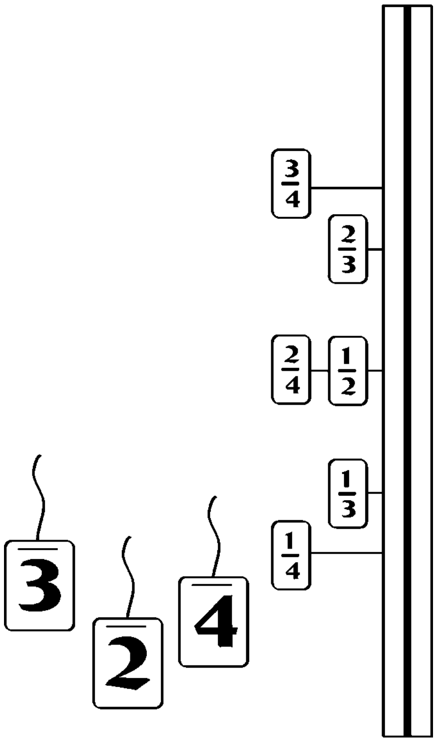

Figure 4. The fraction-lift, with 2-lift, 3-lift and 4-lift. The fractions 1/2 and 2/4 are at the same level.

Figure 5. ‘Room-mates’ of 1/3 and 2/3.

these ‘room-mates’ of 2/3 can be used to find fractions between 1/3 and 2/3 (Figure 5). Here again we see the metaphor of “sharing a room” in connection with positioning fractions on the number line (cf. Sfard, 1994). As 4/12 is a room-mate of 1/3 and 8/12 is one of 2/3, the fractions 5/12, 6/12, etc. are between 1/3 and 2/3.

In her individual work following this class discussion, Audrey shows at least three different strategies to compare the fractions involved:

- she compares the fractions ‘by the look of it’,
- she reasons with the size of the pieces, e.g. 1/3 is bigger than 1/4,
- she reasons with equivalent fractions.

Figure 6. A fragment of the screen of 'treasure-digging'. The fraction 2/5 is found.

In the twentieth lesson we present the students with a computer game, 'treasure-digging'.[4] Here the students are offered a fraction and next they are invited to look for this fraction by clicking on the number line. Every attempt reveals the fraction at the indicated position (Figure 6). This way the students are offered 'anchor-points' to assist them in the searching process.[5]

Audrey plays the game with Ines. Their fourth task is to find the fraction 3/8. In doing so they first dig up the fraction 2/5. Audrey uses the position of this fraction to construct the position of 3/8. She therefore estimates the distance to 1/2, by the look of it. Next, she uses the equivalence of 1/2 and 4/8 to indicate where 3/8 should be: "To the left of 2/5." Audrey thus shows how she uses 1/2 as an reference point to compare 2/5 and 3/8. Audrey's use of the fraction 1/2 as a whole number corresponds with findings of Hunting (1986) and Hart (1981). Both Hart and Hunting state that students relatively easily extend whole numbers to 1/2. Kieren, Nelson and Smith (1985) emphasise the importance of this kind of findings in learning fractions, as the fraction 1/2 can support fraction generating activities. In a similar way, when looking for 9/10 in the tenth game, Audrey shows she is able to compare fractions such as, 3/4, 5/6 and 7/8 by referring to the distance between each fraction and 1.

After twenty one lessons (seven months after the start of her fraction programme) we interview Audrey a second time. If we examine Audrey's increasing ability to compare fractions, on the whole, we observe that Audrey gradually developed various approaches. At first her strategies were restricted to unit-fractions. She compared these fractions by reasoning about the relation between the size of the denominator and that of the fraction. Later she extended her approach to non-unit-fractions. Moreover she learned how to compare fractions by using both 1/2 and 1 as anchor-points. Finally using equivalent fractions was seen as a means of comparing fractions. After about twenty lessons in fractions, Audrey was about to conquer these formal relations between fractions.

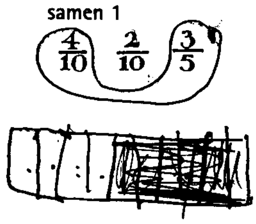

Figure 7. Make 1. Audrey finds equivalent fractions by halving parts.

4.4. *Learning formal fractions*

However, reaching a full understanding of formal fractions is a long process. From the start of the programme Audrey explores formal relations between fractions. At first, these relations are closely connected to situational contexts. For example in the seventh lesson, where fractions emerge when a part of a lighthouse is to be painted. When asked what part still needs to be painted, the students need to find the complement of fractions. Moreover, in this context fractions with denominator 8 are constructed from fractions with denominator 2 and 4, when students are asked to fold the bar-like lighthouse, for example, to 'paint' 5/8. Audrey here explains how she constructed 5/8: "You have to fold in two and again in two and again in two." There are then eight pieces and Audrey knows five of those are needed to make 5/8.

In the following lessons Audrey frequently shows how she relates fractions with denominator 2, 4 and 8. However, it takes some time to extend this knowledge to other fractions. In the fourteenth lesson we observe this for the first time. We ask Audrey to select two fractions from 4/10, 2/10 and 3/5 that together make 1. As Audrey finds this difficult, we advise her to make a sketch. When she does so, she discovers the equivalence of 4/10 and 2/5 (Figure 7).

In the next lessons Audrey repeatedly finds equivalent fractions by doubling both the numerators and the denominators of the fractions. Sometimes she also uses other strategies. For example in the seventeenth lesson we present the students the fraction-lift in the form of a computer-game. Here students use the lifts to move fractions through the building. In one of their games Audrey and Ines need to shift a fraction from 1/5 to 1/3. Audrey suggests the multiplication tables of 3 and 5 can be used to find an appropriate lift. She thus soon finds that the 15-lift can be used.

In the twenty second lesson we introduce 'difficult problems' and 'easy problems' for adding and subtracting fractions. 'Easy problems' are those problems, where the denominators of the fractions that need to be added or subtracted are equal, as with 4/6 + 1/6. In 'difficult problems' the deno-

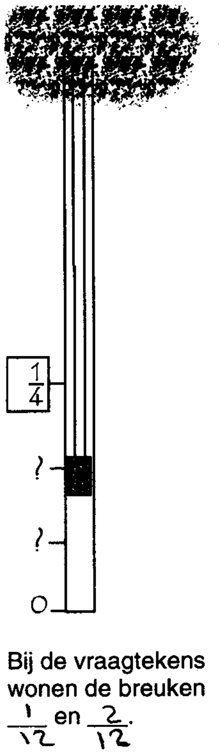

Figure 8. Searching for fractions that "live" at the question-marks. Audrey finds 1/12 and 2/12.

minators of the fractions are unequal, as with 1/3 + 1/2. We discuss with the students how we can turn 4/6 + 1/6 into a 'difficult problem'. The students seem eager to do so. Soon the fraction 4/6 is turned into 8/12 and 1/6 into 6/36. We ask Audrey to suggest still another fraction to replace 4/6 or 1/6. Audrey chooses to change 4/6 into 16/24, by doubling numerator and denominator of 8/12, as mentioned a little earlier.

Later in that lesson Audrey abandons the approach of doubling or halving both numbers in the fraction, to transform the fractions into equivalent ones. For example, she replaces the "easy problem" 5/15 + 3/15 by the "difficult" one 1/3 + 1/5.

Some time later, in the twenty eighth lesson, Audrey shows how she uses equivalent fractions. In this lesson we again present the fraction-lift context. We now ask the students to use this context to divide fractions by two, three or more. When doing this, we see Audrey struggling with the problem of dividing 1/4 by three. She wonders what could be done here. We advise her to search for fractions that are 'at the same floor of the fraction building'. Audrey does so and finds an answer (Figure 8).

We consider formal arithmetic with fractions to be the ability to use equivalent fractions in a proper manner, as equivalent fractions facilitate fraction operations. When we thus analyse Audrey's grasp of equivalent fractions, we see that, initially, manipulating bars results in only one strategy of obtaining equivalent fractions. In the beginning she tends to only double numerator and denominator. However, when the situation forces Audrey to use other equivalence relations, she does so. However, this causes her some difficulty. After 30 lessons Audrey still needs some assistance in finding a general approach to find and use equivalent fractions. If, subsequently, Audrey uses this general strategy, equivalent fractions are used in a creative and flexible manner.

4.5. *Overview*

We conducted a case study on Audrey's fraction learning and found various signs of acquired 'numeracy in fractions' clearly related to the experimental programme and its teaching. Moreover, we found Audrey's flexible strategies in managing equivalent fractions. In this overview, we outline a few of the observations which typify Audrey's growing "numeracy in fractions".

We observed how Audrey needed only a few lessons to fully grasp unit-fractions. Moreover, reasoning with unit-fractions made her develop a way to compare fractions, by considering the denominator. Soon she found more relations between fractions. For example, in the seventh lesson, she constructed 3/8 by repeated halving and in her first interview she easily related 2/5 and 3/5. Later on we saw that her knowledge in fractions is sufficient for her to be able to compare fractions with 1. Moreover, she showed creative use of the fraction 1/2 in various situations.

When we consider Audrey's formal reasoning with fractions, we see that her preferred strategy was the doubling of numerator and denominator. After 30 lessons in fractions she still needed some support in using other approaches to find equivalent fractions. However, when she found an appropriate equivalent fraction, she was proficient in using this fraction.

5. Conclusions and discussion

One of the limitations of a case-study – like the one we described here – is its difficulty to obtain generalisable results. Yin (1984), however, provides a tool to gain some generalisation, namely by explaining case study events in a theoretical framework. That is what we did in the case study described. We first showed how we selected Audrey from her group. Being an average student, Audrey sets an example of how ordinary students can gain proficiency in formal fractions. In line with Yin's (1984) ideas in performing case study research, we ordered our information on Audrey along observed signals of causal relations between the constructed program, and the teaching thereof, and Audrey's learning-process. We thus found that the programme and the teaching thereof could be a factor in Audrey's observed growth in 'numeracy in fractions' and her proficiency in using equivalent fractions.

Simon (1995) provides us with another means to further analyse Audrey's progress. As in our study, Simon performed the roles of both teacher and researcher. He stated that he based his local teaching decisions on the assumed learning of the students. "A hypothetical learning trajectory provides the teacher with a rationale for choosing a particular instructional design; thus, I make my design decisions based on my best guess of how learning might proceed." (Simon, 1995, p. 135). We now used this decision-making design in teaching in analysing key-elements in Audrey's formal fraction learning process, in order to gradually reconstruct the development of Audrey's fraction learning.

We have drawn on Greeno's (1991) notion of situated knowledge to develop an environment that fostered classroom discussion on fraction meanings and relations in order to develop number sense within the fraction domain. We constructed the first activities in the programme so that a language of fractions would be elicited from the students. We therefore used problems where the students had to divide objects. This resulted in Audrey using unit-fractions in a proper way. However, Audrey scarcely used other than unit-fractions here. For that reason we introduced the bar as a measuring instrument. In the activities the bar presented a length of several parts. We expected that the students would now turn to other than unit-fractions, as the measuring activities would give rise to counting. And that is what we observed with Audrey, for example, when she constructed 7/8 as being seven pieces of 1/8. Next, we aimed our teaching activities at developing a number line for fractions. We anticipated the developed bar to serve as a model for fractions. In the situational contexts we used here the number line became a measuring scale on a bar. We observed how this made Audrey shift from the bar to the line and vice versa, for instance when 'hitting' on the try-your-strength machine.

We expected that considering fractions as parts of folded bars or points on a number line would encourage students to compare fractions on several levels. Namely, laying two bars side by side, would give a way of comparing the constructed fractions visually and folding bars would generate a few simple relations between fractions, like 1/2 = 2/4, 3/4 = 6/8, et cetera. Next well chosen situations were developed to elicit other strategies of comparing fractions, such as comparing with 1 and with 1/2. We observed how Audrey soon acquired several fraction comparison strategies. However, we initially observed Audrey having problems in comparing the equivalent fractions 1/3 and 2/6. We therefore used the fraction-lift to clarify equivalent fractions, by making them fractions living at the same floor of the fraction-building and by introducing the metaphor of "roommates" for fractions at the same position on the number line. We now observed how Audrey used the strategy of doubling both the numerator and the denominator to generate equivalent fractions, for example by replacing 2/3 by 4/6 to compare the latter fraction with 5/6.

Folding bars, as we mentioned, cleared the way for the construction of equivalent fractions. We saw how this resulted in Audrey relating fractions with denominators 2, 4 and 8. Moreover, Audrey developed doubling both numerator and denominator as an approach to make equivalent fractions. This turned out to be her favoured (and persisting) strategy. We concluded that by using the bar as a manipulative tool we actually were encouraging this approach (cf. Gravemeijer, 1994). To overcome this one-sided strategy in obtaining equivalent fractions, we introduced several problems, where other approaches were needed, such as constructing a "difficult problem" for the sum 3/15 + 5/15. Moreover we again used the fraction-lift to construct other relations between fractions, for example dividing 1/4 by three. We observed how Audrey slowly started to consider other approaches than doubling both numerator and denominator to construct equivalent fractions.

Simon's (1995) idea of constructing hypothetical learning trajectories as mini-theories of the learning of a student, offered us a useful instrument to analyse Audrey's learning of fractions. Looking at Audrey's progress in gaining proficiency in fractions, the projected learning trajectories became the means of observing and valuing the learning. Teaching interventions following from the analyses were essential in this scheme. They provided possibilities to follow Audrey's progress over an extended period of time and to draw conclusions concerning the potential of Audrey, and that of students like her, to learn formal fractions.

Audrey is not a special student in any aspect, and consequently her learning activities teach us about average students like her. Therefore this study supports the view that the teaching we described, where developing the number line takes a special place and where comparing fractions forms a natural introduction to formal reasoning, could offer many students the prospect of learning formal fractions in a meaningful manner.

NOTES

1. We used the standardized LVS-tests (Janssen, Kraemer and Noteboom, 1995) to obtain the general mathematical skills of the student. Audrey's pre-test score shows she performs at the nation-wide average.
2. Padberg (1989) in an inventory study summarizes all possible approaches for fractions in relation to the four operations: addition, subtraction, division and multiplication.
3. The fraction-lift was an idea of Adrian Treffers.
4. This program was created by Frans van Galen, who earlier made a similar program with whole-number tasks. One of the authors, Ronald Keijzer, proposed to extend the existing program to fractions.
5. The program only shows fractions that facilitate reasoning on several levels. Therefore only fractions with denominators 2, 3, 4, 5, 6, 8, 9, 10 and 12 become visible.

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